J. Fluid Mech. (1979), *vol.* 90, *part* 2, *pp.* 289–303
Printed in Great Britain

Laminar boundary-layer reattachment in supersonic flow

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(Received 8 June 1978)

A rational theory is developed to describe the reattachment of a laminar shear layer in supersonic flow. In the neighbourhood of reattachment the flow develops a three-tiered or 'triple-deck' structure analogous to that which occurs at a point of separation (Stewartson & Williams 1969) and, as in the separation problem, the local flow pattern may be found independently of the flow in the surrounding regions. The fundamental problem of the reattachment triple deck reduces to the solution of the incompressible boundary-layer equations in the lower deck, which is of streamwise and lateral dimensions $O(R^{-\frac{3}{8}})$ and $O(R^{-\frac{5}{8}})$, where $R \gg 1$ is a representative Reynolds number for the flow. Pressure variations in this region are $O(R^{-\frac{1}{4}})$. Asymptotic solutions in terms of x, the scaled streamwise lower-deck variable, are derived to confirm the transition from a reverse flow profile at $x = 0+$, through reattachment, to a forward flow as $x \to \infty$, the attainment of the required asymptotic form downstream (as $x \to \infty$) being shown to depend crucially upon the correct choice of the finite part of the pressure in the lower deck at $x = 0+$. The lower-deck solution is singular at $x = 0+$ and assumes a complicated multi-structured form which is shown to match upstream with the solution in a largely inviscid region of dimension $O(R^{-\frac{1}{2}})$ where the pressure is $O(1)$ and the major part of the flow reversal takes place. Solutions are presented for reattachment at a wall and for symmetric reattachment behind a wedge or bluff body. In the former case the results also explain the apparent ignorance of upstream conditions in the expansive triple-deck solution formulated by Stewartson (1970) in the context of supersonic flow around a convex corner.

1. Introduction

The first complete and rational description of boundary-layer separation was provided by Stewartson & Williams (1969). It was shown that the boundary layer formed along a wall by a uniform supersonic stream can spontaneously separate from the wall through a mechanism which allows a free interaction between the boundary layer and the external flow. The 'triple-deck' region in which this interaction takes place has lateral and streamwise length scales $O(\epsilon^3 l)$, where $\epsilon \ll 1$ is defined by

$$\epsilon^{-8} = R = U_\infty l/\nu_\infty \tag{1.1}$$

and R is the Reynolds number of the flow based on a convenient length scale l and the velocity and kinematic viscosity of the external stream, denoted by U_∞ and ν_∞ respectively. The triple deck divides laterally into three decks: an upper deck of height $O(\epsilon^3 l)$ which is inviscid and irrotational, a main deck of the same height $O(\epsilon^4 l)$ as the boundary layer and a lower deck of height $O(\epsilon^5 l)$. The fundamental problem reduces to the solution of the incompressible boundary-layer equations in the lower deck, and the numerical solution by Williams (1975) has provided a precise value of the pressure rise through separation, which is of order ϵ^2.

The importance of the triple deck in the theory of high Reynolds number flow is now without question. The assertion by Sychev (1972) that a similar asymptotic structure also governs separation in incompressible flow is strongly supported by the recent numerical calculations of Smith (1977). Many other examples of triple-deck applications, some but by no means all of which involve separation, include flows due to injection (Smith & Stewartson 1973), flows at corners (Stewartson 1970) and flows at trailing edges (Stewartson 1969; Messiter 1970).

In the present paper it is suggested that a triple-deck structure also governs the flow at reattachment and that, as in the original separation problem, the flow pattern in the triple deck can be found independently of the detailed structure and geometry of the overall flow. Specific models to which the theory should apply, and which have been widely studied, for example in experiments by Chapman, Kuehn & Larson (1958), Hama (1968) and Batt & Kubota (1968), include flows behind wedges, behind backward-facing steps and on compression ramps. Theoretical investigations of base flow by Denison & Baum (1963) and Weiss (1967) have been improved by Burggraf (1970) and more recently by Messiter, Hough & Feo (1973), who considered the flow behind a backward-facing step and included a detailed analysis of the reattachment region which indicated the possibility of a triple-deck region downstream. Their theory was concerned with a step whose height is small compared with the boundary-layer length l, so that the major portion of the separated shear layer is of Blasius form to a first approximation and reattachment occurs within a short distance of the step. The return flow which feeds the separated shear layer is provided by a backward jet which emanates from a region of streamwise extent $O(\epsilon^4 l)$ at the point where the shear layer reattaches to the wall. The flow in this region is governed by inviscid equations and the return of the pressure to its ambient value downstream fixes the length of the separated shear layer since, by Bernoulli's equation, the pressure rise across the small inviscid region must be balanced by the increase in velocity along the dividing streamline of the shear layer. This leads to an inviscid flow in which reattachment is not quite completed, suggesting, as in the previous study by Burggraf (1970), that the reattachment point is asymptotically far downstream of the inviscid zone.

In § 2 it is argued that the flow reversal in the inviscid zone must, under quite general circumstances, have the property that reattachment is almost, but not quite, completed. An asymptotic solution is provided which supports this assertion and also indicates a division of the inviscid zone into three separate layers downstream which match precisely with the lateral subdivisions of the triple deck. The triple-deck structure is formulated in § 3, the major difference from the separation triple deck being that the appropriate scaled streamwise variable x is now restricted to the semi-infinite range $0 < x < \infty$, the inviscid zone where the shear layer first strikes the wall being sited at $x = 0$.

The remainder of the paper considers the solution of the fundamental problem in the lower deck of the triple deck. The asymptotic solution as $x \to 0+$ involves a subdivision of the lower deck into three regions of lateral extent $O(x)$, $O(x|\log x|^{\frac{1}{2}})$ and $O(x^{-1})$, the presence of the second region, slightly wider than the first, being crucial in determining the correct asymptotic form. A similar structure is also relevant in the convex-corner problem studied by Stewartson (1970) and is discussed in § 6. The importance of a correct understanding of the asymptotic structure at $x = 0+$ is that the procedure by which the lower-deck equations must be solved then becomes apparent, for we find that the constant p_1 in the expansion of the pressure at $x = 0+$ is arbitrary and must be chosen to give the correct asymptotic form downstream (as $x \to \infty$) to match with the boundary-layer flow beyond reattachment. The solution as $x \to \infty$ is discussed in §4.2 and flux considerations confirm that the expansion is consistent with that at $x = 0+$ and therefore consistent with the reattachment of the flow at some finite value of x, x_R say. An approximate method is devised to obtain crude estimates of the values of both x_R and p_1. In our basic model we consider reattachment at an impermeable wall although the modifications which occur in the case of two shear layers which merge symmetrically behind a wedge or bluff body are considered briefly in §5.

2. The inviscid zone

We shall suppose that a shear layer of thickness $O(\epsilon^4 l)$ separates an external supersonic stream from a region of stagnant flow and strikes a wall at the origin O of a set of Cartesian axes Ox^*y^*, where x^* and y^* are in directions along and normal to the wall respectively. If we denote the corresponding velocity components by u^* and v^*, the pressure by p^*, the density by ρ^* and consider the flow in the region where X and Y, defined by

$$x^* = \epsilon^4 X, \quad y^* = \epsilon^4 Y, \tag{2.1}$$

are of order one, we have

$$u^* = U(X, Y) + \ldots, \quad v^* = V(X, Y) + \ldots, \quad p^* = P(X, Y) + \ldots, \quad \rho^* = R(X, Y) + \ldots, \tag{2.2}$$

where, since $\epsilon \ll 1$, U, V, P and R satisfy the inviscid equations

$$\left.\begin{aligned} R\left(U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right) = -\frac{\partial P}{\partial X}, \quad R\left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) = -\frac{\partial P}{\partial Y}, \\ \frac{\partial}{\partial X}(RU) + \frac{\partial}{\partial Y}(RV) = 0, \quad R\left(U\frac{\partial P}{\partial X} + V\frac{\partial P}{\partial Y}\right) = \gamma P\left(U\frac{\partial R}{\partial X} + V\frac{\partial R}{\partial Y}\right). \end{aligned}\right\} \tag{2.3}$$

Here γ is the ratio of specific heats and the temperature has been eliminated in favour of the density using the equation of state. The boundary conditions for these equations include specification of the incoming shear-layer profile upstream, the external velocity at $Y = \infty$ and the inviscid wall condition $V = 0$ at $Y = 0$. The slip velocity $U(X, 0)$ is then reduced to zero at the wall within a viscous sublayer of height $O(\epsilon^6 l)$ (see figure 1). Although a complete solution of (2.3) is not possible, the general structure of the inviscid flow must conform to one of three possible patterns corresponding to the behaviour $Y = Y_s(X)$ of the dividing streamline of the incoming shear layer. First, we may have $Y_s = 0$ at $X = X_R$, in which case reattachment is completed within the inviscid region and we may expect a forward flow profile at $X = \infty$, with $U(\infty, 0) > 0$. However, by a form of Bernoulli's equation obtainable from (2.3) along $Y = 0$, this would imply that the wall pressure $P(X, 0)$ reaches a maximum at the stagnation point $X = X_R$ and then decreases downstream and this seems improbable. Second, we may have $Y_s > 0$ for all $-\infty < X < \infty$, in which case the velocity profile at $X = \infty$ is reversed at the wall, $U(\infty, 0) < 0$. But this seems equally unlikely since the inviscid

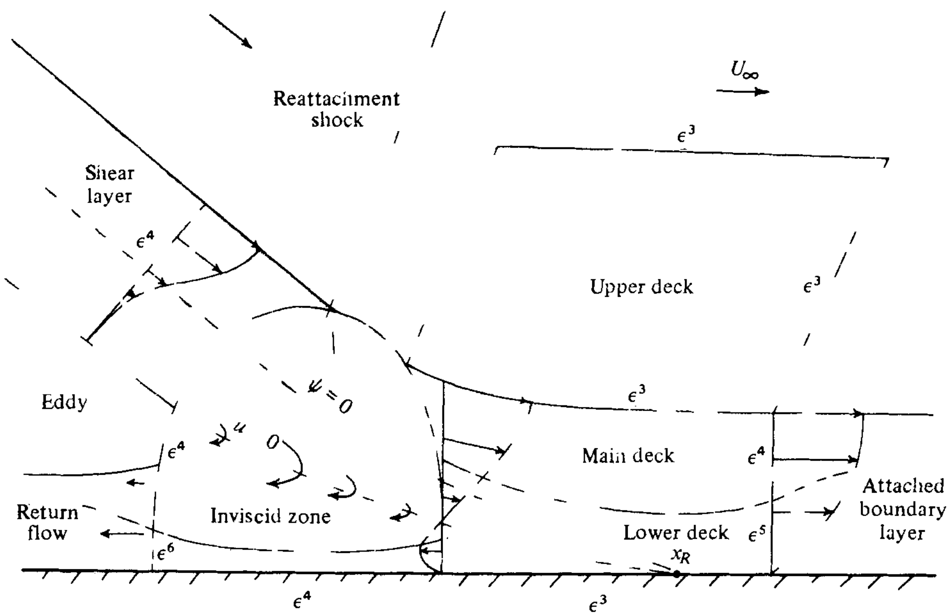

FIGURE 1. A schematic diagram of the flow regions near reattachment.

zone would then serve no real purpose and in all probability the boundary conditions for the flow outside this zone would be underspecified. The remaining possibility is that $Y_s \to 0$ as $X \to \infty$, in which case $U(\infty, 0) = 0$, so that reattachment is nearly, but not quite, completed. The wall pressure $P(X, 0)$ now increases monotonically through the inviscid zone from its value in the stagnant flow upstream to the ambient value p_∞ as $X \to \infty$, where the following asymptotic structure may be developed for $Y = O(1)$:

$$\left.\begin{aligned} P &\to p_\infty + \frac{C}{X^2} + \dots, \quad R \to R_0(Y) + \frac{D}{X} R_0'(Y) + \dots \\ U &\to U_0(Y) + \frac{D}{X} U_0'(Y) + \dots, \quad V \to \frac{D}{X^2} U_0(Y) + \dots \end{aligned}\right\} \quad (X \to \infty). \tag{2.4}$$

The precise form of the velocity and density profiles U_0, and R_0 will depend upon the details of the flow upstream but, as advanced above, we assume that $U_0(0) = 0$ and also $U_0(\infty) = U_\infty$, where U_∞ is the velocity in the external flow. We shall define

$$U_0'(0) = \lambda, \quad R_0(0) = R_w \quad \text{and} \quad R_0(\infty) = R_\infty.$$

Then since (2.3) gives $RU\partial U/\partial X = -\partial P/\partial X$ on $Y = 0$ we have

$$R_w \lambda^2 D^2 = -2C. \tag{2.5}$$

Further, in the region where $\zeta = Y/X = O(1)$ we have

$$P \sim p_\infty + \frac{C}{X^2}[1 - \zeta(M_\infty^2 - 1)^{\frac{1}{2}}]^{-2}, \quad V \sim \frac{C(M_\infty^2 - 1)^{\frac{1}{2}}}{X^2 R_\infty U_\infty}[1 - \zeta(M_\infty^2 - 1)^{\frac{1}{2}}]^{-2} \quad [\zeta < (M_\infty^2 - 1)^{-\frac{1}{2}}], \tag{2.6}$$

where $M_\infty^2 = U_\infty^2 R_\infty/\gamma p_\infty$, and matching with the solutions (2.4) gives

$$C(M_\infty^2 - 1)^{\frac{1}{2}}/R_\infty U_\infty = DU_\infty. \tag{2.7}$$

Equations (2.5) and (2.7) now determine the unknown constants C and D, so that

$$U \to U_0(Y) - \frac{2U_\infty^2 R_\infty U_0'(Y)}{XR_w \lambda^2 (M_\infty^2-1)^{\frac{1}{2}}} + \dots, \quad P \to p_\infty - \frac{2U_\infty^4 R_\infty^2}{X^2 R_w \lambda^2 (M_\infty^2-1)} + \dots \quad (X \to \infty). \tag{2.8}$$

There seems no reason to suppose that the eigenfunctions corresponding to (2.8) are absent, in which case we should have $C = D = 0$.

3. The reattachment triple deck

The flow leaving the inviscid zone enters a triple deck in which streamwise variations occur on the slightly longer length scale $x_1 = O(1)$, where

$$x^* = \epsilon^3 x_1 \quad (0 < x_1 < \infty). \tag{3.1}$$

The triple deck divides laterally into three decks. In the main deck, $Y = O(1)$ and

$$\left.\begin{aligned} u^* &= U_0(Y) + \epsilon A_1(x_1)\, U_0'(Y) + \dots, \quad v^* = -\epsilon^2 U_0(Y)\, A_1'(x_1) + \dots, \\ \rho^* &= R_0(Y) + \epsilon A_1(x_1)\, R_0'(Y) + \dots, \quad p^* = p_\infty + \epsilon^2 P_1(x_1) + \dots \end{aligned}\right\} \quad (\epsilon \to 0). \tag{3.2}$$

Here $A_1(x_1)$ is a function of x_1 which represents the displacement thickness of the boundary layer. This is related to the pressure P_1 through the solution of the wave equation in the upper deck, where $Y_1 = \epsilon Y = O(1)$ and

$$p^* = p_\infty + \epsilon^2 P_1[x_1 - Y_1(M_\infty^2 - 1)^{\frac{1}{2}}]. \tag{3.3}$$

The lower deck, where $Y = O(\epsilon)$, is required to reduce the slip velocity at the base of the main deck to zero and here the final problem reduces to the solution of the incompressible boundary-layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} \tag{3.4}$$

subject to the boundary conditions

$$u = v = 0 \quad (y = 0), \tag{3.5}$$

$$u \sim y - A(x) \quad (y \to \infty), \quad \text{where} \quad p = A'(x), \tag{3.6}$$

$$p \sim -2/x^2 \quad (x \to 0+), \tag{3.7}$$

$$p \to 0 \quad (x \to \infty). \tag{3.8}$$

These equations and boundary conditions have been reduced to their simplest form through the transformations

$$x_1 = ax, \quad Y = \epsilon b y, \quad P_1 = cp, \quad A_1 = bA, \quad u^* = \epsilon d b^{-1} u, \quad v^* = \epsilon^3 d a^{-1} v, \tag{3.9}$$

where

$$\left.\begin{aligned} a &= \left(\frac{M_\infty^2}{(M_\infty^2-1)^{\frac{1}{2}}}\right)^{\frac{3}{4}} \left(\frac{\epsilon^8}{\nu_w}\right)^{\frac{1}{4}} \frac{\lambda^{\frac{1}{4}}}{\lambda_1^{\frac{3}{2}}}, \quad b = \left(\frac{M_\infty^2}{(M_\infty^2-1)^{\frac{1}{2}}}\right)^{\frac{1}{4}} \left(\frac{\nu_w}{\epsilon^8}\right)^{\frac{1}{4}} \lambda^{-\frac{1}{4}} \lambda_1^{-\frac{1}{2}}, \\ c &= \frac{\gamma p_\infty M_\infty \lambda_1}{(M_\infty^2-1)^{\frac{1}{4}} \lambda^{\frac{1}{2}}} \left(\frac{\nu_w}{\epsilon^8}\right)^{\frac{1}{2}}, \quad d = \frac{M_\infty \lambda^{\frac{1}{2}}}{(M_\infty^2-1)^{\frac{1}{4}} \lambda_1} \left(\frac{\nu_w}{\epsilon^8}\right)^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.10}$$

ν_w is the kinematic viscosity at the wall and $\lambda_1 = M_0'(0)$, where $M_0^2(Y) = R_0 U_0^2/\gamma p_\infty$ is the local Mach number in the boundary layer.

The full derivation of the system (3.4)–(3.10) including details of matching between the three decks is given by Stewartson & Williams (1969); the only differences here are that the system is restricted to the range $0 < x < \infty$ instead of $-\infty < x < \infty$ and the pressure must satisfy the boundary conditions (3.7) and (3.8). The first of these ensures that the triple-deck solution matches with that in the inviscid zone upstream, the main- and upper-deck solutions matching with (2.8) and (2.6) respectively. The second condition ensures that the triple-deck solution matches with an attached boundary-layer flow downstream and is discussed in §4.3 below.

Of the parameters $\lambda = U_0'(0)$, $\lambda_1 = M_0'(0)$ and ν_w which appear in (3.10), the last is dependent upon the viscosity law chosen. The first two depend upon the details and geometry of the flow upstream, and although their values are not required to solve the lower-deck problem, we note that for the specific case of a backward-facing step approximate values may be inferred from the results of Messiter *et al.* (1973) as

$$\lambda = KU_\infty l^{-1}[1+\tfrac{1}{2}(\gamma-1)M_\infty^2]^{-1}, \tag{3.11}$$

$$\lambda_1 = KM_\infty l^{-1}[1+\tfrac{1}{2}(\gamma-1)M_\infty^2]^{-\frac{3}{2}}. \tag{3.12}$$

These formulae are based on the assumptions that the step height is τl, where $\tau \ll 1$, that the wall is adiabatic and that the viscosity varies linearly with temperature. l is taken as the length of the Blasius boundary layer upstream of the step and reattachment is at a distance $O(\tau^{\frac{3}{2}}l)$ downstream of the step. The numerical constant K is related to the precise form of the profile in the separated shear layer as it enters the inviscid zone.

We emphasize that the reattachment problem (3.4)–(3.8) is a universal one, formulated quite independently of any of the assumptions related to the above model, and it now remains to demonstrate that a consistent solution of the lower-deck equations may be found which incorporates reattachment at some finite value of x.

4. The lower-deck problem

From (3.4) we introduce the stream function ψ defined by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x; \quad \psi = 0 \quad \text{on} \quad y = 0. \tag{4.1}$$

4.1. *Asymptotic solution as $x \to 0+$: the reverse flow profile upstream*

The leading terms are essentially those given by the terms in the asymptotic expansion of the inviscid-zone solution as $X \to \infty$ so we expect $u = O(x^{-1})$ and $p = O(x^{-2})$. It then follows that viscous effects are significant when $y = O(x)$. More generally, if we consider the possibility that $y = xf(x)\tilde{y}$ and $\psi = f(x)\tilde{\psi}(\tilde{y})$, where $\tilde{y}$ and $\tilde{\psi}$ are of order one, we find that the four terms in the lower-deck momentum equation (3.4) involve orders of magnitude x^{-3}, $x^{-3}f^{-2}$ and $x^{-2}f'f^{-1}$, so that, in addition to $f = 1$, a balance may be possible with $f = |\log x|^{\frac{1}{2}}$. It emerges that the two regions corresponding to both these values of f are present in the asymptotic structure of the lower-deck solution as $x \to 0+$. We shall show that this leads to a pressure expansion of the form

$$p = -2x^{-2}+p_1+x^2\{p_2|\log x|^{\frac{1}{2}}+p_3+\ldots\}+\ldots \quad (x\to 0+), \tag{4.2}$$

with
$$A = 2x^{-1}+p_1 x+x^3\{\tfrac{1}{3}p_2|\log x|^{\frac{1}{2}}+\tfrac{1}{3}p_3+\ldots\}+\ldots \quad (x\to 0+), \tag{4.3}$$

where the coefficient p_1 remains arbitrary.

In the inner region, where $\eta = y/x$ is the order-one variable, the expansion for the stream function has the form

$$\psi = F_0(\eta) + x^2\{G_0(\eta) + |\log x|^{-\frac{1}{2}} G_1(\eta) + |\log x|^{-1} G_2(\eta) + \ldots\}$$
$$+ x^4\{|\log x|^{\frac{1}{2}} H_0(\eta) + H_1(\eta) + \ldots\} + \ldots \quad (x \to 0+). \tag{4.4}$$

Substitution into the lower-deck equation then gives at leading order the system

$$F_0''' + F_0'^2 = 4, \quad F_0(0) = F_0'(0) = 0, \quad F_0'(\infty) = -2, \tag{4.5}$$

where the first two boundary conditions follow from (3.5) and the third from the outer condition (3.6), in which A is given by (4.3). The consistency of the coefficients of the $O(x^{-2})$ and $O(x^{-1})$ terms in p and A (respectively) is a result of the same interaction between the upper and the main decks as that which leads to (2.7). The appropriate solution of (4.5), which is shown in figure 2, is

$$F_0' = 4 - 6\tanh^2(\eta + C_0), \quad C_0 = \tanh^{-1}(\tfrac{2}{3})^{\frac{1}{2}} = 1{\cdot}146\ldots, \tag{4.6}$$

and we have $F_0 \sim -2\eta + C_1$ as $\eta \to \infty$, where $C_1 = 6[1 - (\frac{2}{3})^{\frac{1}{2}}]$.

In the 'transitional' region where

$$\theta = y/x|\log x|^{\frac{1}{2}} \tag{4.7}$$

is of order one, the stream-function expansion begins

$$\psi = -2|\log x|^{\frac{1}{2}}\theta + C_1 + x^2\{|\log x|\, g_0(\theta) + |\log x|^{\frac{1}{2}} g_1(\theta) + g_2(\theta) + \ldots\}$$
$$+ x^4\{|\log x|\, h_0(\theta) + |\log x|^{\frac{1}{2}} h_1(\theta) + h_2(\theta) + \ldots\}$$
$$+ x^4 \exp(2i\theta|\log x|^{\frac{1}{2}})\{|\log x|^{\frac{1}{2}} A_0(\theta) + A_1(\theta) + \ldots\} + \text{c.c.} + \ldots. \tag{4.8}$$

Here the leading terms are simply the continuation of the unchanged inner solution F_0 and at order x^4 we find (below) that oscillatory terms with spatial frequency

$$O(|\log x|^{\frac{1}{2}}) \quad \text{as} \quad x \to 0+$$

must be included in the expansion, the complex conjugate being denoted by c.c.

The equation for g_0 is

$$g_0''' + \theta g_0'' - g_0' = 0, \tag{4.9}$$

but since we require $g_0' \sim \theta + 0$ as $\theta \to \infty$, and, in order to match with the inner solution, $g_0(0) = g_0'(0) = 0$, the solution is simply

$$g_0 = \tfrac{1}{2}\theta^2. \tag{4.10}$$

Although a term $O(|\log x|^{\frac{1}{2}})$ in the pressure would result in a constant p_{10}, say, on the right-hand side of (4.9) we should then require $g_0' \sim \theta + 2p_{10}$ as $\theta \to \infty$, so that, from (4.9), $p_{10} = 0$.

The solution (4.10) now provides the outer boundary condition for G_0 in the system

$$G_0''' + 2F_0'' G_0 = 0, \quad G_0(0) = G_0'(0) = 0, \quad G_0''(\infty) = 1 \tag{4.11}$$

and the numerical solution, shown in figure 2, has the property

$$G_0 \sim \tfrac{1}{2}\eta^2 + C_2\eta + C_3 \quad (\eta \to \infty), \tag{4.12}$$

where $C_2 = -0{\cdot}920$ and $C_3 = 0{\cdot}964$. The constant C_2 now generates the term g_1 in the transitional region, which must satisfy

$$g_1''' + \theta g_1'' = 0, \quad g_1(0) = 0, \quad g_1'(0) = C_2, \quad g_1'(\infty) = -p_1, \tag{4.13}$$

the last condition arising from the outer condition (3.6). The solution is

$$g_1' = C_2 - (p_1 + C_2)\operatorname{erf}(\theta/2^{\frac{1}{2}}), \tag{4.14}$$

where p_1 remains completely arbitrary. At $\theta = 0$ the value of $g_1''(0)$ provides the outer condition for G_1, which then generates g_2 and so on:

$$G_1''' + 2F_0''G_1 = 0, \quad G_1(0) = G_1'(0) = 0, \quad G_1''(\infty) = -2(2/\pi)^{\frac{1}{2}}(p_1 + C_2), \tag{4.15}$$

$$g_2''' + \theta g_2'' + g_2' = 0, \quad g_2(0) = 0, \quad g_2'(0) = \lim_{\eta\to\infty}\{G_1' + 2(2/\pi)^{\frac{1}{2}}(p_1 + C_2)\eta\}, \quad g_2 = o(\log\theta) \quad (\theta\to\infty). \tag{4.16}$$

There will be an infinite set of functions G_i and g_i ($i = 0, 1, 2, \ldots$) associated with successive powers of $|\log x|^{-\frac{1}{2}}$.

In (4.16) the last condition is required for consistency with the solution outside the transitional layer, where the order-one variable is

$$\xi = yx. \tag{4.17}$$

Strictly speaking we should apply the condition (3.6) as $\xi \to \infty$ for the first two terms in the expansion of u outside the inner region are $-2/x$ and y; these are of the same order of magnitude when $\xi = O(1)$, and indeed as $\xi \to \infty$ it is the second which dominates the first. Fortunately the solution in this outer region does not really affect the expansion, the appropriate terms being continuations of the inner solutions and at most quadratic functions of ξ. The expansion is

$$\psi = x^{-2}(\tfrac{1}{2}\xi^2 - 2\xi) + (C_1 - p_1\xi) + x^2\{|\log x|^{\frac{1}{2}}(\tfrac{5}{3}p_2 - \tfrac{1}{3}p_2\xi) + (\tfrac{5}{3}p_3 - \tfrac{1}{3}p_3\xi) + \ldots\} + \ldots, \tag{4.18}$$

where the last two terms match with the transitional-layer solution as $\xi \to 0$ provided that

$$p_2 = \tfrac{3}{5}\pi^{-\frac{1}{2}}(p_1 + C_2), \quad p_3 = \tfrac{3}{5}g_2(\infty), \tag{4.19}$$

successive terms being generated by the finite parts of the $g_i(\theta)$ as $\theta \to \infty$.

It is of interest to note that at this stage in the expansion there is no requirement that the transitional layer be present, although then we must take $p_1 = -C_2$, in which case the solution is completely determinate, a fact commented upon by Stewartson (1970) in his study of a related triple-deck singularity arising in convex-corner flows. However the presence of the transitional layer is confirmed beyond doubt by the higher-order terms in the expansion. In the inner region the functions H_0 and H_1 are generated by p_2, p_3 and F_1:

$$H_0''' - 2F_0'H_0' + 4F_0''H_0 = 2p_2, \quad H_0(0) = H_0'(0) = 0, \quad H_0 \sim \tfrac{1}{2}p_2\eta \quad (\eta\to\infty), \tag{4.20}$$

$$H_1''' - 2F_0'H_1' + 4F_0''H_1 = 2p_3 + F_1'^2 - 2F_1F_1'', \quad H_1(0) = H_1'(0) = 0, \quad H_1 \sim \tfrac{1}{2}p_3\eta \quad (\eta\to\infty). \tag{4.21}$$

We see that since $F_0' \to -2$ ($\eta\to\infty$) two of each of the three complementary solutions of these equations are oscillatory as $\eta\to\infty$. It follows that, if the boundary conditions

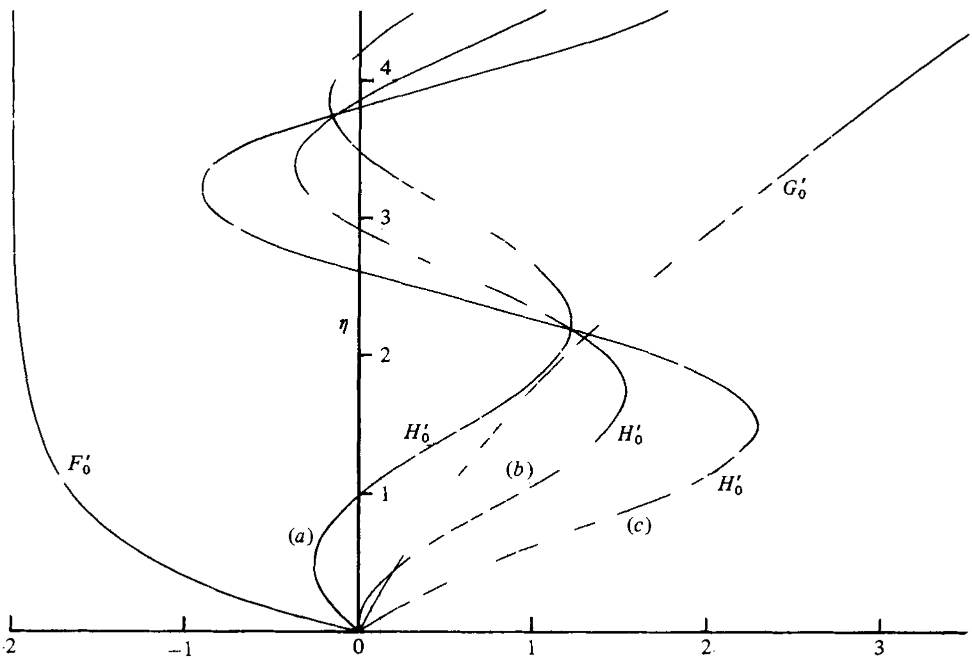

FIGURE 2. Solutions for the inner-region functions F_0', G_0' and H_0'. The three solutions for H_0' correspond to (a) $\Delta_0 = -1$, (b) $\Delta_0 = 0$ and (c) $\Delta_0 = 1$, where $\Delta_0 = H_0''(0)$ and $p_2 = 1$.

at the wall are to be satisfied, oscillations are forced into the outer regions of the flow, in particular

$$H_0 \sim p_2\{\tfrac{1}{2}\eta + \alpha_0(\Delta_0)\cos 2\eta + \beta_0(\Delta_0)\sin 2\eta + \gamma_0(\Delta_0)\} \quad (\eta \to \infty), \tag{4.22}$$

where we may suppose that the numerical solution of (4.20) will provide values of α_0, β_0 and γ_0 as functions of $\Delta_0 = H_0''(0)$. Three such solutions are shown in figure 2.

In the transitional layer the non-oscillatory part of H_0 automatically matches with the solution

$$h_0' = \tfrac{1}{2}\theta g_1' - \tfrac{1}{2}g_1 + \tfrac{1}{2}p_2 - \tfrac{1}{4}\theta^2 g_1'', \tag{4.23}$$

which, since $h_0' \sim -\frac{1}{2}\pi^{-\frac{1}{2}}(p_1 + C_2)$ as $\theta \to \infty$, also matches automatically with the outer solution (4.18). The oscillatory part of H_0 generates the function A_0, which satisfies the equation

$$A_0' + \tfrac{1}{3}\theta A_0 = 0 \tag{4.24}$$

and therefore assures the decay of the oscillations, with

$$A_0 = \tfrac{1}{2}p_2(\alpha_0 - i\beta_0)\exp(-\tfrac{1}{6}\theta^2). \tag{4.25}$$

The corresponding solution for A_1 is

$$A_1 = \tfrac{1}{2}p_3(\alpha_1 - i\beta_1)\exp(-\tfrac{1}{6}\theta^2) - \tfrac{1}{3}i\theta(\alpha_0 - i\beta_0)(1 - \tfrac{1}{18}\theta^2)\exp(-\tfrac{1}{6}\theta^2), \tag{4.26}$$

where α_1 and β_1 are functions of $\Delta_1 = H_1''(0)$ and are obtained from the solution of (4.21). We see that, in addition to p_1, the full expansion as $x \to 0+$ also involves an infinite set of indeterminate constants Δ_i $(i = 0, 1, 2, \ldots)$; further unknown constants arise from oscillations of higher frequency generated by all the higher-order terms in the inner expansion in powers of x^2 and $|\log x|^{-\frac{1}{2}}$.

The major properties of the upstream profile are now determined. Viscous effects, including weak oscillations, are confined to two layers of thickness $O(x)$ and $O(x|\log x|^{\frac{1}{2}})$. The inner one matches, as $x \to 0+$, with the viscous sublayer of the inviscid zone of thickness $O(\epsilon^6)$ while the outer one also indicates the presence of a slightly wider sublayer of thickness $O(\epsilon^6|\log \epsilon|^{\frac{1}{2}})$ between the inviscid zone and the wall. In the outer region where $\xi = O(1)$ the stream function is

$$\psi \sim \tfrac{1}{2}y^2 - 2y/x + \ldots \quad (x \to 0+), \tag{4.27}$$

so that the line of zero streamwise velocity, $y \sim 2/x$, bisects the region between the wall and the dividing streamline, $y \sim 4/x$. Outside this line the forward flow profile increases to the uniform shear prescribed by the boundary-layer profile $U_0(Y)$ at $Y = 0$. The singular part of the pressure matches with that in the inviscid zone, while the value of p_1 will provide an $O(\epsilon^2)$ correction to the inviscid pressure upstream. The determination of p_1 is discussed in §4.3 below.

4.2. *Asymptotic solution as $x \to \infty$: the forward flow profile downstream*

The requirement that $p \to 0$ as $x \to \infty$ leads to an asymptotic structure of the same form as that derived by Smith & Stewartson (1973) in the context of slot injection into a laminar boundary layer. Thus we expect

$$\psi = \tfrac{1}{2}y^2 + f_1(\eta_1) + \ldots, \quad p = \alpha x^{-\frac{4}{3}} + \ldots, \quad A = -3\alpha x^{-\frac{1}{3}} + \ldots \quad (x \to \infty), \tag{4.28}$$

where $\eta_1 = y/x^{\frac{1}{3}}$ and

$$f_1' = \frac{3^{\frac{4}{3}}\alpha}{(-\frac{2}{3})!}\int_0^{\eta_1} \exp\left(-\tfrac{1}{9}\eta_1^3\right) d\eta_1. \tag{4.29}$$

The constant α remains undetermined by the lower-deck equations and boundary conditions but may be related to the constant p_1 by general considerations of mass flux. From §4.1 the flux into the lower deck between the points $y = 0$ and $y = y_0$ ($\gg 1$) on the line $x = x_0$ ($\ll 1$) is

$$\int_0^{y_0} \left(-\frac{2}{x} + y - p_1 x\right) dy + \int_0^{\infty} (F_0' + 2)\, d\eta \sim \tfrac{1}{2}y_0^2 - \frac{2y_0}{x_0} + C_1. \tag{4.30}$$

The flux entering through $y = y_0$ in the region $x_0 < x < \infty$ is

$$-\int_{x_0}^{\infty} v(x, y_0)\, dx = -[yA - A' - \tfrac{1}{2}A^2]_{x_0}^{\infty} \sim \frac{2y_0}{x_0} - 3p_1, \tag{4.31}$$

and since (4.31) and (4.30) must be balanced by the flux

$$\int_0^{y_0} (y + x^{-\frac{1}{3}}f_1')\, dy \sim \tfrac{1}{2}y_0^2 + f_1(\infty) \tag{4.32}$$

leaving through $x = \infty$ we obtain from the finite parts

$$\alpha = -\frac{(-\frac{2}{3})!}{3^{\frac{5}{3}}(-\frac{1}{3})!}(C_1 - 3p_1). \tag{4.33}$$

This confirms that the upstream profile obtained in §4.1 is consistent with the attainment of a forward profile downstream and thus with the reattachment of the flow at a finite value of x given by x_R, where $0 < x_R < \infty$.

Downstream of the triple deck ($x^* > 0$) the forward flow along the wall is simply found by solving the boundary-layer equations subject to the initial profile

$$u^* = U_0(Y) \quad \text{at} \quad x^* = 0$$

and the boundary condition $u^* \to U_\infty$ as $Y \to \infty$. We expect the compressible Blasius profile to evolve as $x^* \to \infty$.

4.3. *The numerical solution of the lower-deck problem*

The expansions of §4.1 and 4.2 now point the way to the correct procedure for the full solution of the triple-deck reattachment problem. Previous studies of the numerical solution of the lower-deck equations based on finite-difference schemes have shown that the solution will, in general, evolve into one of three possible asymptotic structures as x increases (see Stewartson 1974). In the original problem of Stewartson & Williams (1969) for instance, an initial positive pressure increment leads to the compressive solution, in which separation occurs and the pressure p reaches a plateau value of about 1·8 as $x \to \infty$. An initial negative increment leads to the expansive solution, which terminates in a singularity at a finite value of x, while if the pressure increment is zero, the Blasius solution ($u = y$, $v = p = 0$) simply continues unchanged. Similar solutions are discussed in relation to the flow at the trailing edge of a flat plate by Daniels (1974).

In the present problem a similar situation must arise and the attainment of the correct asymptotic form (4.28) downstream will depend upon the correct choice of the arbitrary constant p_1 in the expansion of the pressure at $x = 0+$. If p_1 is too large we may expect the compressive solution to evolve as $x \to \infty$ and the pressure eventually to attain a plateau value. If p_1 is too small the expansive solution will evolve and the solution will terminate in a singularity. The required value of p_1 is the one which separates these two possibilities. Its numerical determination will be a formidable task, for not only do we have to contend with the singular structure of the solution at $x = 0+$, but the presence of large negative velocities will necessitate iterative backward sweeps over the reverse flow region from the reattachment point possibly using an extension of the boundary-layer scheme of Klemp & Acrivos (1972).

In view of these difficulties we shall for the present resort to an approximate method of solution. A simple patching of the two asymptotic solutions yields encouragingly realistic results, mainly because of the close similarity of the expansions (4.2) and (4.28). We require the displacement thickness and pressure to be continuous at $x = x_p$, so that

$$-2/x_p^2 + p_1 = \alpha x_p^{-\frac{4}{3}}, \quad 2/x_p + p_1 x_p = -3\alpha x_p^{-\frac{1}{3}}. \tag{4.34}$$

The resulting cubic for p_1 has just one real root,

$$p_1 = 0{\cdot}032 \quad (\text{with} \quad x_p = 5{\cdot}569, \quad \alpha = -0{\cdot}318),$$

and the corresponding solutions are shown in figure 3. Continuity of A at $x = x_p$ ensures that the discontinuity in velocity profile at this point is restricted to the neighbourhood of the wall; a few profiles for more realistic values of x are included in figure 3. The estimate of the reattachment position based on the first two terms of the small x expansion (4.4) is

$$x_R = 2{\cdot}753. \tag{4.35}$$

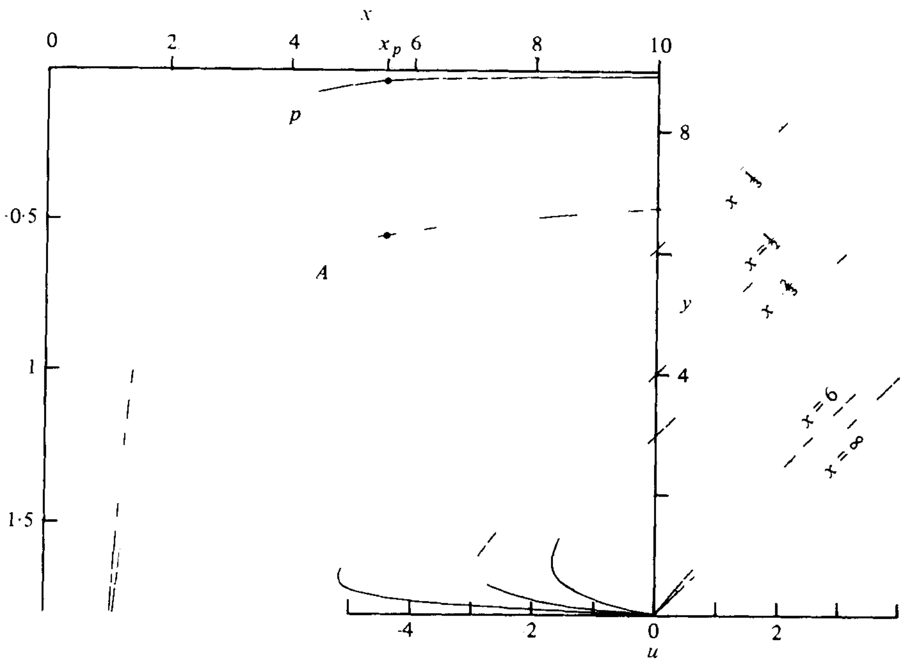

FIGURE 3. Curves of pressure, displacement function and velocity profiles based on the approximation (4.34) and computed from the asymptotic solutions with $p_1 = 0{\cdot}032$, $x_p = 5{\cdot}569$ and $\alpha = -0{\cdot}318$ (small x expansion for $x < x_p$; large x expansion for $x > x_p$).

5. Reattachment behind a wedge

If the reattachment takes the form of two symmetric free shear layers which merge behind a wedge at $x^* = y^* = 0$, the previous formulation of the inviscid zone and triple-deck structure is unchanged except that the lower-deck boundary condition (3.5) is now replaced by

$$v = \partial u/\partial y = 0 \quad \text{on} \quad y = 0 \quad (0 < x < \infty). \tag{5.1}$$

The main difference in the asymptotic structure at $x = 0+$ is that the inner viscous region is no longer required. The transitional-region solution now has the form

$$\psi = -2\theta|\log x|^{\frac{1}{2}} + x^2\{|\log x|\, g_1(\theta) + |\log x|^{\frac{1}{2}} g_2(\theta)\} + x^4\{|\log x|^{\frac{3}{2}} h_{01}(\theta) + O(|\log x|)\} + \dots \quad (x \to 0+), \tag{5.2}$$

where
$$g_1' = \theta\, \mathrm{erf}\,(\theta/2^{\frac{1}{2}}) + (2/\pi)^{\frac{1}{2}} \exp\,(-\tfrac{1}{2}\theta^2), \quad g_2 = -p_1\theta \tag{5.3}$$

and
$$h_{01}' = \tfrac{1}{4}(g_1'^2 - 2g_1 g_1'' + \tfrac{3}{5} \times 2^{\frac{1}{2}}). \tag{5.4}$$

We find that the constant p_1 in the expansion of the pressure is again arbitrary and the leading-order correction to this term is now $O(|\log x|^{\frac{1}{2}})$ larger than before:

$$p = -2/x^2 + p_1 + x^2\{\tfrac{3}{10} \times 2^{\frac{1}{2}}|\log x| + O(|\log x|^{\frac{1}{2}})\} + \dots \quad (x \to 0+). \tag{5.5}$$

In the absence of the wall the reverse flow region simply extends almost linearly across the region between the lines $y \sim \pm 2/x$, the maximum reverse velocity being $u \sim -2/x$ on the line of symmetry $y = 0$. Although the inner viscous region is virtually redundant, with effectively $F_0' = -2$, we note that the oscillatory solutions are not necessarily ruled out since (4.20) and (4.21) now have complementary eigensolutions of the form $\sin 2\eta$ which obey the boundary conditions (5.1).

Downstream, the appropriate expansions in the lower deck as $x \to \infty$ are

$$\psi = x^{\frac{2}{3}} f_0(\eta_1) + 3\alpha\eta_1 + \ldots, \quad p = p_0 x^{-\frac{2}{3}} + \alpha x^{-\frac{4}{3}} + \ldots, \tag{5.6}$$

the leading term being that which occurs in the trailing-edge problem studied by Stewartson (1970), with

$$f_0''' + \tfrac{2}{3} f_0 f_0'' - \tfrac{1}{3} f_0'^2 = 0, \quad f_0(0) = f_0''(0) = 0 \quad \text{and} \quad f_0' \sim \eta_1 - 3p_0 \quad (\eta_1 \to \infty).$$

The numerical solution gives $p_0 = -0{\cdot}297$. Once again the coefficient α is determined in terms of p_1 by flux considerations. In this case we obtain

$$\alpha = -p_1/3p_0. \tag{5.7}$$

Downstream of the triple deck the problem in the wake is to solve the boundary-layer equations subject to the double-structured initial profile supplied by $U_0(Y)$, where $Y = O(1)$ and by f_0 where $Y/x^{*\frac{1}{3}} = O(1)$. The solution now approaches the uniform profile U_∞ as $x^* \to \infty$. The reattachment position is now interpreted as $x = x_R$, where $u(x_R, 0) = 0$ and the estimate based on the expansion (5.2) is $x_R = 1{\cdot}806$.

6. Discussion

We have obtained a consistent picture of laminar boundary-layer reattachment for supersonic flow in the limit as the Reynolds number tends to infinity. The basic structure is similar to the triple-deck structure which describes separation and involves an interaction between the boundary layer and the inviscid flow. Just upstream of reattachment the majority of the flow reversal takes place in an inviscid zone of dimension $O(R^{-\frac{1}{2}})$ in which the dividing streamline approaches the wall only asymptotically, and final reattachment is completed in the lower deck of streamwise extent $O(R^{-\frac{3}{8}})$. Our approximate solution of the lower-deck equations suggests that the position of reattachment is at a distance

$$x_R^* = 2{\cdot}753 R^{-\frac{3}{8}} \left(\frac{M_\infty^4}{M_\infty^2 - 1}\right)^{\frac{3}{8}} \left(\frac{\epsilon^8}{\nu_w}\right)^{\frac{1}{4}} \lambda^{\frac{1}{4}} \lambda_1^{-\frac{3}{2}} \tag{6.1}$$

downstream of the point where the centre-line of the shear layer intersects the wall. Quantitative comparisons with experimental work will require a full numerical solution of the lower-deck equations but we note that the overall flow pattern and monotonic pressure variation near reattachment are in agreement with the experimental evidence available. For reattachment at a wall we have shown that the final pressure decay downstream is proportional to $R^{-\frac{1}{4}}(R^{\frac{3}{8}}x^*/l)^{-\frac{5}{3}}$ while for a symmetric reattachment this is replaced by the slower decay $R^{-\frac{1}{4}}(R^{\frac{3}{8}}x^*/l)^{-\frac{2}{3}}$. Another contrast is in the variation of $A(x)$ as $x \to \infty$, a more rapid narrowing of the boundary layer just downstream of reattachment being predicted in the symmetric case.

One of the interesting features of the lower-deck solution is the presence of weak spatial oscillations of an asymptotically high frequency. Matching implies that these will persist in the flow upstream and their possible absence in the symmetric problem suggests that physically they may be the manifestation of a weak instability associated with the region of large shear near the wall. Also we note that the presence of the transitional layer in the lower-deck expansion at $x = 0+$ resolves the apparent absence of an indeterminacy of the expansive triple-deck solution discussed by Stewartson (1970) in the context of convex-corner flow. Here the singularity arises as x increases and the basic solution outlined by Stewartson has

$$p \sim -2/\tilde{x}^2, \quad u \sim y + 2/\tilde{x} \quad (y \gg \tilde{x}, \quad \tilde{x} \to 0+), \tag{6.2}$$

where $\tilde{x} = -x$. The expansion now represents an entirely forward flow accelerating to a sink-like singularity at $\tilde{x} = 0$. The results of §4.1 for the higher-order terms in the expansion may be applied directly to this problem simply by changing the signs of η, θ and ξ in the various equations and expansions. The constant p_1 is thus again arbitrary and its indeterminacy now represents the ignorance of conditions upstream of the singularity. In fact it seems likely that this expansion will also be relevant to the problem of a backward-facing step in the region at the top of the step. Here a pressure variation $O(1)$ must occur in an inviscid zone (I_1) of dimension $O(R^{-\frac{1}{2}})$, governed by the system (2.3), where the backward jet which originates from the inviscid zone at reattachment (I_2) mixes with the oncoming boundary layer. Just upstream of the step a triple-deck solution with $p^* - p_\infty = O(R^{-\frac{1}{4}})$ will terminate in the singularity (6.2) and match with the solution in the inviscid zone I_1 (in essentially the same way as in the reattachment problem; §§3 and 4). The appropriate value of p_1, the leading-order correction to the pressure in (6.2), was computed with a finite-difference scheme, marching downstream from an initial profile $u = y$ with $p = -10^{-5}$. In contrast to the reattachment problem, there are no difficulties concerning reverse flow; the numerical results reported by Stewartson (1970) were recovered and a closer scrutiny of the singularity gave

$$p_1 \simeq -0{\cdot}3. \tag{6.3}$$

In specific problems there are still various features of the laminar high Reynolds number flow which are not adequately understood. One is the lip shock which stems from the region I_1 referred to above and which has been studied experimentally by Hama (1968). In the outer flow this reacts with the main reattachment shock from the zone I_2 considered in §2. In the wake flow further study is also required to determine the properties of the recirculating eddy enclosed by the detached shear layer and backward jet upstream of reattachment.

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